Int. J. Multiphase Flow Vol. 13, No. 6, pp. 837–844, 1987
Printed in Great Britain.

0301-9322/87 $3.00 + 0.00

THE FUNCTIONAL SIMILARITY BETWEEN FAXÉN RELATIONS AND SINGULARITY SOLUTIONS FOR FLUID–FLUID, FLUID–SOLID AND SOLID–SOLID DISPERSIONS

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(*Received* 30 *December* 1986; *in revised form* 15 *May* 1987)

Abstract—The functional similarity between Faxén relations (for moments of the appropriate transport flux) and singularity solutions has been noted in the past. For rigid particles (and perfect conductors etc.) it has been noted in the literature that the root of this similarity is linked to the Lorentz reciprocal theorem. However, the duality applies even to more general two-phase problems such as a viscous drop in another solvent, with the relevant singularity distribution taken from the exterior solution. Although two-phase Faxén relations are available for various particle shapes, until now, the root of this duality has not been demonstrated explicitly. The application of the duality is illustrated by the derivation of new Faxén relations for ellipsoidal inclusions.

1. INTRODUCTION

Many interesting particle interaction problems have not been considered rigorously because the hydrodynamics, at least as expressed in the classical works, appear intractable. Typically, rigorous models are developed only for spheres and the results are extrapolated to nonspherical shapes by the use of empirical shape factors. Despite the availability of supercomputers, three-dimensional flow in an unbounded fluid region, a central problem of microhydrodynamics of suspensions, is still beyond the reach of "direct" numerical methods such as finite elements and finite differences.

One possible approach is to divide the particle–particle interaction domain into "long-range" and "near-neighbor" regions. The near-neighbor region is bounded but requires an appropriate numerical solution (one possibility will be presented in a subsequent paper). The long-range region obviously occupies most of the parameter space, but fortunately, even for nonspherical particles, this region can be treated analytically by asymptotic methods such as the "method of reflections". It must be emphasized here that explicit examples of the "method of reflections", as developed in standard texts such as Happel & Brenner (1973), use special properties of spheres, such as the addition theorem for spherical harmonics, and are not directly applicable to other shapes. However, an alternate but equivalent approach based on a multipole expansion of disturbance fields combined with the use of the Faxén relations for the coefficients is now available. (The method is equivalent for spheres because Lamb's general solution is equivalent to a multipole expansion and the Faxén laws are equivalent to addition theorems). It is interesting that Faxén derived his well-known relations for the force and torque on a sphere by manipulation of Lamb's general solution and thus his method cannot be readily generalized. Derivation of the Faxén relations for particles of arbitrary shape was first done by Brenner (1964) using the Lorentz reciprocal theorem, and all subsequent works in this area (e.g. Rallison 1978) have followed this approach.

In filtration theory, the capture efficiency is one of the key model parameters. The dividing trajectory which delineates the capture region can be determined quite accurately from the hydrodynamics of the long-range region. The analyses described by Spielman (1977) can be readily extended to nonspherical shapes such as fibers and disks, with the Faxén relations playing a central role in determining the effect of particle orientation in the upstream region, on the dividing trajectory. The Faxén relations also play an important part in the development of effective constitutive equations for the transport properties of multiphase media (Batchelor 1974).

Faxén (1924) showed that the force and torque on a rigid sphere of radius a in an unbounded fluid of viscosity μ with an ambient flow $\mathbf{v}^{\infty}$ are given by

$$\mathbf{F} = 6\pi\mu a\left(1 + \frac{a^2}{6}\nabla^2\right)\mathbf{v}^{\infty}(\mathbf{0}) \quad \text{and} \quad \mathbf{T} = 4\pi\mu a^3 \nabla \times \mathbf{v}^{\infty}(\mathbf{0}),$$

respectively. Extensions of Faxén's original relations include expressions for higher-order moments such as the stresslet (Batchelor & Green 1972), expressions for ellipsoids (Brenner 1964; Rallison 1978), expressions for a viscous spherical drop (Hetsroni & Haber 1970; Rallison 1978) and expressions for the heat output and (thermal) dipoles in the analogous heat conduction problems (Brenner & Haber 1984; Haber & Brenner 1984; O'Brien 1979).

One intriguing aspect of the Faxén relations is the functional similarity between them and certain singularity solutions, i.e. solutions for a specific boundary condition (the so-called conjugate field) expressed in terms of the Green's function. This duality has been noted in the past by Hinch (1977) and for rigid particles (and thus also for perfect conductors) its origin is readily demonstrated as shown by Kim (1985). However, this duality appears even in two-phase problems, such as Stokes flow past a viscous drop suspended in a different fluid or a particle imbedded in a matrix of a different thermal conductivity. For example, the Faxén law for the force on a spherical drop of viscosity μ_2 and radius a in a second fluid of viscosity μ_1 is (Hetsroni & Haber 1970)

$$\mathbf{F} = 4\pi\mu_1 a\frac{2+3\lambda}{2+2\lambda}\left[1 + \frac{\lambda}{2(3\lambda+2)}a^2\nabla^2\right]\mathbf{v}^{\infty}(\mathbf{0}), \qquad [1]$$

where $\lambda = \mu_2/\mu_1$. On the other hand, the exterior solution for the same drop in a uniform stream $\mathbf{U}$ can be constructed by the singularity method as a combination of the Stokes monopole (or stokeslet) and potential dipole located at the sphere center, i.e.

$$\mathbf{v} = \mathbf{U} - 4\pi\mu_1 a\frac{2+3\lambda}{2+2\lambda}\left[1 + \frac{\lambda}{2(3\lambda+2)}a^2\nabla^2\right]\mathbf{U}\cdot\frac{\mathbf{I}(\mathbf{x})}{8\pi\mu_1}. \qquad [2]$$

The divergence of the associated stress field may be written as

$$\nabla\cdot\boldsymbol{\sigma} = 4\pi\mu_1 a\frac{2+3\lambda}{2+2\lambda}\left[1 + \frac{\lambda}{2(3\lambda+2)}a^2\nabla^2\right]\mathbf{U}\delta(\mathbf{x}). \qquad [3]$$

The functional similarity between [1] and [3] is striking for it suggests the not so obvious conclusion that even in two-phase problems, the functional form of the Faxén law may be extracted given only the form taken by the exterior solution. *It should also be noted that none of the prior derivations in the literature offer an explanation for the source of this duality.*

There are two reasons for pursuing this matter further. The first reason is pedagogical. Students notice this duality and naturally, want to know whether it is a fluke or whether one can derive it directly. The proof is straightforward (two lines) for rigid particles but, as will be seen presently, quite involved for viscous drops. A second reason is that singularity solutions are more general than solutions by separable-coordinate systems. Thus, using the duality, one can develop Faxén relations to solve hydrodynamic interactions for particles of quite complex shape. As a case in point, we note that singularity solutions are known for geometries such as the point force outside a sphere, so one can develop Faxén relations for rod–sphere assemblies and solve exactly the hydrodynamics of filtration by spherical collectors.

In the subsequent sections, we show explicitly (for the general situation of arbitrary particle shape, no knowledge of the interior solution and for both Stokes flow and heat conduction) that this duality originates because the Faxén relations are always the result of an integration involving generalized functions associated with the distribution of singularities of the exterior solution of the conjugate boundary-value problem.

2. THE HEAT CONDUCTION PROBLEM

2.1. The general procedure

We consider first the disturbance induced by the placement of a single particle of thermal conductivity k_2 in a matrix of conductivity k_1. The ambient field will be denoted by $T^{\infty}(\mathbf{x})$. Steady heat conduction is governed by Laplace's equation and it proves convenient to start with an integral representation for the temperature in which the double-layer potential has been eliminated:

$$T_{\text{out}}(\mathbf{x}) = \frac{1}{4\pi}\frac{k_1 - k_2}{k_2}\int_S (\mathbf{n}\cdot\nabla T_{\text{out}})\frac{1}{|\mathbf{x}-\mathbf{x}_s|}\,dS(\mathbf{x}_s) \qquad \text{if } \mathbf{x}\in\text{matrix},$$

$$T_{\text{in}}(\mathbf{x}) = \frac{1}{4\pi}\frac{k_1 - k_2}{k_2}\int_S (\mathbf{n}\cdot\nabla T_{\text{out}})\frac{1}{|\mathbf{x}-\mathbf{x}_s|}\,dS(\mathbf{x}_s) \qquad \text{if } \mathbf{x}\in\text{particle}. \qquad [4]$$

The multipole expansion of [4] requires knowledge of the multipole moments,

$$\frac{k_2 - k_1}{k_2}\int_{S_p} \mathbf{q}\cdot\mathbf{n}\,\mathbf{x}_s\mathbf{x}_s\ldots\mathbf{x}_s\,dS(\mathbf{x}_s).$$

The first moment

$$\mathbf{S} = \frac{k_2 - k_1}{k_2}\int_{S_p} \mathbf{q}\cdot\mathbf{n}\,\mathbf{x}_s\,dS(\mathbf{x}_s)$$

is the *thermal dipole* and appears in the theories for the effective conductivity of the two-phase medium (Jeffrey 1973). Our objective is to show that the functional form of the Faxén relation for the dipole may be obtained from the singularity solution of the exterior field for the particle in the linear temperature field $\mathbf{G}\cdot\mathbf{x}$.

The first part of the development follows the lines presented in O'Brien (1979). We start with Green's second identity applied to the region between the particle and a large surface S_{∞} enclosing the particle:

$$\int_{S_{\infty}} (T_1\mathbf{q}_2 - T_2\mathbf{q}_1)\cdot\mathbf{n}\,dS = \int_{S_p} (T_2\mathbf{q}_1 - T_1\mathbf{q}_2)\cdot\mathbf{n}\,dS, \qquad [5]$$

where T_1 and T_2 are two temperature fields and $\mathbf{q}_1$ and $\mathbf{q}_2$ are the associated heat fluxes. We let

$$T_1(\mathbf{x}) = T_l(\mathbf{x}) - \mathbf{G}\cdot\mathbf{x} \qquad [6a]$$

and

$$T_2(\mathbf{x}) = T(\mathbf{x}) - T^{\infty}(\mathbf{x}), \qquad [6b]$$

where T_l denotes the temperature field for the particle placed in the linear field $\mathbf{G}\cdot\mathbf{x}$ and T is the (unknown) temperature field that results when the particle is placed in the ambient field $T^{\infty}(\mathbf{x})$.

The integral over S_{∞} vanishes in the limit as S_{∞} is expanded to infinity because of the decay in the temperature and flux fields. In addition, the exterior fields in the surface integral over S_p may be replaced with the interior fields, by using the continuity in the temperature and flux fields. (Note that gradients of kT_l, kT, $\mathbf{G}\cdot\mathbf{x}$ and T^{∞} are continuous.) Thus in terms of *interior* fields,

$$\int_{S_p} (T_2\mathbf{q}_1 - T_1\mathbf{q}_2)\cdot\mathbf{n}\,dS = 0, \qquad [7]$$

with $\mathbf{q}_1 = -k_2\nabla T_l + k_1\mathbf{G}$ and $\mathbf{q}_2 = -k_2\nabla T + k_1\nabla T^{\infty}$.

We now apply the divergence theorem to obtain

$$\int_{V_p} (\nabla T_2\cdot\mathbf{q}_1 - \nabla T_1\cdot\mathbf{q}_2)\,dV = 0, \qquad [8]$$

so that upon substitution of the expressions for T_1 and T_2,

$$(k_1 - k_2)\int_{V_p} \nabla T\cdot\mathbf{G}\,dV = (k_1 - k_2)\int_{V_p} \nabla T_l\cdot\nabla T^{\infty}\,dV. \qquad [9]$$

The l.h.s. of this equation yields the dipole so that

$$\mathbf{S}\cdot\mathbf{G} = (k_1 - k_2)\int_{V_p} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V. \qquad [10]$$

Now at this point, if the *interior* solution is available, one takes the obvious route by evaluating the r.h.s. of [10] [as done by O'Brien (1979) for the sphere]. However, this obvious approach misses the more general statement.

We may expand the r.h.s. of [10] as follows:

$$\begin{aligned}(k_1 - k_2)\int_{V_p} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V &= k_1\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V - k_2\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V \\ &= k_1\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V - k_2\int_{V_p^-} \nabla\cdot(\nabla T_I\, T^\infty)\, \mathrm{d}V \\ &= k_1\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V - k_2\int_{S_p^-} \mathbf{n}\cdot\nabla T_I\, T^\infty \, \mathrm{d}S \\ &= k_1\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V - k_1\int_{S_p^+} \mathbf{n}\cdot\nabla T_I\, T^\infty \, \mathrm{d}S \\ &= k_1\int_{V_p^-} \nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V - k_1\int_{V_p^+} \nabla\cdot(\nabla T_I\, T^\infty)\, \mathrm{d}V \\ &= -k_1\int_{V_p^+} \nabla^2 T_I T^\infty \, \mathrm{d}V - k_1\left(\int_{V_p^+} - \int_{V_p^-}\right)\nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V.\end{aligned}$$

The notation V_p^- and V_p^+ indicates explicitly whether the interior or exterior solution is employed. Note that the last two volume integrals cancel each other because

$$\begin{aligned}\left(\int_{V_p^+} - \int_{V_p^-}\right)\nabla T_I\cdot\nabla T^\infty \, \mathrm{d}V &= \left(\int_{V_p^+} - \int_{V_p^-}\right)\nabla\cdot(T_I\cdot\nabla T^\infty)\, \mathrm{d}V \\ &= \left(\int_{S_p^+} - \int_{S_p^-}\right)\mathbf{n}\cdot(T_I\cdot\nabla T^\infty)\, \mathrm{d}S = 0.\end{aligned}$$

(The gradient of T^∞ is continuous across S_p.) Thus [10] may be rewritten as

$$\mathbf{S}\cdot\mathbf{G} = -k_1\int_{V_p^+} \nabla^2 T_I\, T^\infty \, \mathrm{d}V. \qquad [11]$$

Now if the exterior solution is written as a singularity solution, then $\nabla^2 T_I$ becomes the corresponding distribution of singularities, $\mathbf{G}\cdot\nabla\delta(\mathbf{x}-\boldsymbol{\xi})$ where $\delta(\mathbf{x})$ is the Dirac delta function. An integration by parts of this generalized function leads to the same functional of $\nabla T^\infty(\boldsymbol{\xi})$. For example, the exterior solution for a sphere in a linear field may be written as (Jeffrey 1973)

$$T_I = \mathbf{G}\cdot\mathbf{x} + 4\pi a^3\frac{k_2 - k_1}{k_2 + 2k_1}\mathbf{G}\cdot\nabla\frac{1}{4\pi r} \qquad [12]$$

so that

$$\nabla^2 T_I = -4\pi a^3\frac{k_2 - k_1}{k_2 + 2k_1}\mathbf{G}\cdot\nabla\delta(\mathbf{x}). \qquad [13]$$

The Faxén relation for the dipole follows from [11] as

$$\mathbf{S} = -4\pi a^3\frac{k_2 - k_1}{k_2 + 2k_1}k_1\nabla T^\infty(\mathbf{0}). \qquad [14]$$

2.2. The ellipsoidal inclusion

We now consider an ellipsoidal inclusion of conductivity k_2 with its surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

($a \geqslant b \geqslant c$), in a matrix of conductivity k_1. Given the ambient temperature field $\mathbf{G}\cdot\mathbf{x}$, the exterior solution may be written in the singularity form

$$T = \mathbf{G}\cdot\mathbf{x} - (\mathbf{M}\cdot\mathbf{G})\cdot\nabla\int_{\mathrm{E}} \frac{f(\xi)}{4\pi k_1|\mathbf{x}-\xi|}\,\mathrm{d}\mathbf{A}(\xi), \tag{15}$$

where $\mathbf{M}$ is a second-order, diagonal tensor with elements given by

$$M_{11} = -\frac{8\pi}{3}k_1(k_2-k_1)\left[(k_2-k_1)\int_0^\infty \frac{\mathrm{d}t}{(a^2+t)\Delta(t)} + \frac{2k_1}{abc}\right]^{-1}, \tag{16}$$

with $\Delta(t) = \sqrt{(a^2+t)(b^2+t)(c^2+t)}$ and M_{22} and M_{33} obtained by cycling the dependence on a, b and c. Physically, $\mathbf{M}$ specifies the linear relation between the dipole and the ambient field, i.e., for the linear field, $\mathbf{S} = \mathbf{M}\cdot\mathbf{G}$. The function $f(\xi)$ is the density function for the distribution of the singularities.

As in Stokes flow (Kim 1985) the distribution is over the interior of the focal ellipse $\mathrm{E}(\xi)$, i.e. the ellipse in the plane $z = 0$ with semiaxes a_{E} and b_{E} given by

$$a_{\mathrm{E}}^2 = a^2 - c^2 \quad \text{and} \quad b_{\mathrm{E}}^2 = b^2 - c^2.$$

The density function is defined over the focal ellipse as

$$f(\xi) = f(x, y) = \frac{3q}{2\pi a_{\mathrm{E}} b_{\mathrm{E}}} \quad \text{with} \quad q(x, y) = \sqrt{1 - x^2/a_{\mathrm{E}}^2 - y^2/b_{\mathrm{E}}^2}.$$

From the duality we obtain the following Faxén relation for the dipole of an ellipsoidal inclusion:

$$\mathbf{S} = \mathbf{M}\cdot\int_{\mathrm{E}} f(\xi)\,\nabla T^\infty(\xi)\,\mathrm{d}\mathbf{A}(\xi). \tag{17}$$

As shown in Kim & Arunachalam (1987), the integral over the focal ellipse may be recast as an expansion in Brenner's (1964) operator $D^2 = a^2\partial/\partial x^2 + b^2\partial/\partial y^2 + c^2\partial/\partial z^2$ using the identity

$$\frac{2n-1}{2\pi a_{\mathrm{E}} b_{\mathrm{E}}}\int_{\mathrm{E}} q^{2n-3}\varphi(\xi)\,\mathrm{d}\mathbf{A}(\xi) = \frac{(2n)!}{2^n n!}\left[\left(\frac{1}{D}\frac{\partial}{\partial D}\right)^{n-1}\frac{\sinh D}{D}\right]\varphi(\mathbf{0})$$

which holds for any harmonic function $\varphi(\xi)$.† Therefore, the Faxén relation may also be written as

$$\mathbf{S} = 3\mathbf{M}\cdot\left[\left(\frac{1}{D}\frac{\partial}{\partial D}\right)\frac{\sinh D}{D}\right]\nabla T^\infty(\mathbf{0}). \tag{18}$$

Although we have restricted our discussion to the dipole, this reciprocity also applies to higher order moments and the singularity solution for

$$T^\infty = H_{k_1 k_2 \ldots k_n} x_{k_1} x_{k_2} \ldots x_{k_n}.$$

3. THE FAXÉN LAW FOR VISCOUS DROPS

The derivation of the Faxén relations for a viscous drop is quite analogous to that shown in the previous section for the heat conduction problem. The temperature field, heat flux and Green's second identity are replaced respectively by the velocity field, stress field and the Lorentz reciprocal theorem (Happel & Brenner 1973). The essential ideas will be demonstrated by deriving the Faxén relation for the stresslet and the force. The derivation presented here should be compared with that of Hetsroni & Haber (1970) and Rallison (1978).

We consider the disturbance induced by the placement of a viscous Newtonian drop of viscosity μ_2 in a Newtonian fluid of viscosity μ_1. We take both fluids to be incompressible. The ambient velocity field will be denoted by $\mathbf{v}^\infty(\mathbf{x})$. The governing equations are the Stokes equation

$$-\nabla p + \mu\nabla^2\mathbf{v} = 0 \tag{19}$$

and the equation of continuity,

$$\nabla\cdot\mathbf{v} = 0. \tag{20}$$

†The operator on the r.h.s. of the preceding equation is defined symbolically by its power series in D^2.

We will derive the Faxén relation for the stresslet,

$$\mathbf{S}=\int_{V_p}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_f)\mathrm{d}V=\frac{1}{2}\int_{S_p}[(\boldsymbol{\sigma}\cdot\mathbf{n}\mathbf{x}_s+\mathbf{x}_s\boldsymbol{\sigma}\cdot\mathbf{n})-2\mu_1(\mathbf{v}\mathbf{n}+\mathbf{n}\mathbf{v})]\,\mathrm{d}S(\mathbf{x}_s), \qquad [21]$$

where the first equality is the fundamental definition of the stresslet as the difference of the actual stress and that obtained using the constitutive law of the exterior fluid. The stresslet is the key suspension mechanical quantity in the expression for the effective viscosity of the two-phase medium as reviewed by Batchelor (1974). We will show that the functional form of the Faxén relation may be obtained from the singularity solution of the exterior field for the drop in the linear velocity field $\mathbf{E}\cdot\mathbf{x}$. We start with the Lorentz reciprocal theorem applied to the region between the particle and the large surface S_∞ enclosing the particle:

$$\int_{S_\infty}(\mathbf{v}_1\cdot\boldsymbol{\sigma}_2\cdot\mathbf{n}-\mathbf{v}_2\cdot\boldsymbol{\sigma}_1\cdot\mathbf{n})\,\mathrm{d}S=\int_{S_p^+}(\mathbf{v}_1\cdot\boldsymbol{\sigma}_2\cdot\mathbf{n}-\mathbf{v}_2\cdot\boldsymbol{\sigma}_1\cdot\mathbf{n})\,\mathrm{d}S, \qquad [22]$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two velocity fields that satisfy the governing equations [19] and [20] and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the associated stress fields, i.e. $\boldsymbol{\sigma}=-p\boldsymbol{\delta}+2\mu_1\mathbf{e}$. We will use the notation $\mathbf{e}=(\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}})/2$ for the rate-of-strain field.

We let

$$\mathbf{v}_1(\mathbf{x})=\mathbf{v}_l(\mathbf{x})-\mathbf{E}\cdot\mathbf{x} \qquad [23]$$

and

$$\mathbf{v}_2(\mathbf{x})=\mathbf{v}(\mathbf{x})-\mathbf{v}^\infty(\mathbf{x}), \qquad [23]$$

where $\mathbf{v}_l$ denotes the velocity field for the particle placed in the linear field $\mathbf{E}\cdot\mathbf{x}$ and $\mathbf{v}$ is the (unknown) velocity field that results when the particle is placed in the ambient field $\mathbf{v}^\infty(\mathbf{x})$.

As before, the integral over S_∞ vanishes and the exterior fields in the surface integral over S_p may be replaced with the interior fields, with proper attention paid to the jump conditions across the particle surface. All velocity fields and $\mathbf{E}\cdot\mathbf{n}$ and $\mathbf{e}^\infty\cdot\mathbf{n}$ are continuous across S_p, while $\boldsymbol{\sigma}\cdot\mathbf{n}$ and $\boldsymbol{\sigma}_l\cdot\mathbf{n}$ suffer a jump equal to $\gamma\mathbf{n}$ where γ is the surface tension of the fluid–drop interface. Thus in terms of *interior* fields,

$$\int_{S_p^-}(\mathbf{v}_1\cdot\boldsymbol{\sigma}_2\cdot\mathbf{n}-\mathbf{v}_2\cdot\boldsymbol{\sigma}_1\cdot\mathbf{n})\,\mathrm{d}S=0. \qquad [24]$$

Note that there are no contributions from the surface tension term because $\mathbf{v}\cdot\mathbf{n}=0$ on S_p.

We now apply the divergence theorem to obtain

$$\int_{V_p^-}(\nabla\mathbf{v}_1:\boldsymbol{\sigma}_2-\nabla\mathbf{v}_2:\boldsymbol{\sigma}_1)\,\mathrm{d}V=0 \qquad [25]$$

so that upon substitution of the expressions for $\mathbf{v}_1$, $\mathbf{v}_2$, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$,

$$\mathbf{E}:\int_{V_p^-}(\boldsymbol{\sigma}-2\mu_1\mathbf{e})\,\mathrm{d}V=\int_{V_p^-}\mathbf{e}^\infty:(\boldsymbol{\sigma}_l-2\mu_1\mathbf{e}_l)\,\mathrm{d}V. \qquad [26]$$

The l.h.s. of this equation yields the stresslet while the r.h.s. may be rearranged as follows:

$$\begin{aligned}\mathbf{E}:\mathbf{S}&=\int_{V_p^-}\mathbf{e}^\infty:\boldsymbol{\sigma}_l\,\mathrm{d}V-\int_{V_p^-}\mathbf{e}^\infty:2\mu_1\mathbf{e}_l\,\mathrm{d}V\\&=\int_{V_p^-}\nabla\cdot(\mathbf{v}^\infty\cdot\boldsymbol{\sigma}_l)\,\mathrm{d}V-\int_{V_p^-}\mathbf{e}^\infty:2\mu_1\mathbf{e}_l\,\mathrm{d}V\\&=\int_{S_p^-}\mathbf{v}^\infty\cdot(\boldsymbol{\sigma}_l\cdot\mathbf{n})\,\mathrm{d}S-\int_{V_p^-}\mathbf{e}^\infty:2\mu_1\mathbf{e}_l\,\mathrm{d}V\\&=\int_{S_p^+}\mathbf{v}^\infty\cdot(\boldsymbol{\sigma}_l\cdot\mathbf{n})\,\mathrm{d}S-\int_{V_p^-}\mathbf{e}^\infty:2\mu_1\mathbf{e}_l\,\mathrm{d}V,\end{aligned}$$

where again the surface tension term makes no contribution—this time because $\mathbf{v}^\infty$ is a solenoidal field. We continue as follows:

$$\begin{aligned}\mathbf{E}:\mathbf{S} &= \int_{V_p^+} \nabla\cdot(\mathbf{v}^\infty\cdot\boldsymbol{\sigma}_1)\,dV - \int_{V_p^-} \mathbf{e}^\infty:2\mu_1\mathbf{e}_1\,dV \\ &= \int_{V_p^+} \mathbf{v}^\infty\cdot(\nabla\cdot\boldsymbol{\sigma}_1)\,dV + \int_{V_p^+} \mathbf{e}^\infty:\boldsymbol{\sigma}_1\,dV - \int_{V_p^-} \mathbf{e}^\infty:2\mu_1\mathbf{e}_1\,dV \\ &= \int_{V_p^-} \mathbf{v}^\infty\cdot(\nabla\cdot\boldsymbol{\sigma}_1)\,dV + \left(\int_{V_p^+} - \int_{V_p^-}\right)\boldsymbol{\sigma}^\infty:\mathbf{e}_1\,dV.\end{aligned}$$

As in the heat conduction problem, the last two volume integrals cancel each other because

$$\begin{aligned}\left(\int_{V_p^+} - \int_{V_p^-}\right)\boldsymbol{\sigma}^\infty:\mathbf{e}_1\,dV &= \left(\int_{V_p^+} - \int_{V_p^-}\right)\nabla\cdot(\boldsymbol{\sigma}^\infty\cdot\mathbf{v}_1)\,dV \\ &= \left(\int_{S_p^+} - \int_{S_p^-}\right)(\boldsymbol{\sigma}^\infty\cdot\mathbf{n})\cdot\mathbf{v}_1\,dS = 0,\end{aligned}$$

since $\boldsymbol{\sigma}^\infty\cdot\mathbf{n}$ is continuous across S_p. Therefore,

$$\mathbf{E}\cdot\mathbf{S} = \int_{V_p^+} \mathbf{v}^\infty\cdot(\nabla\cdot\boldsymbol{\sigma}_1)\,dV. \qquad [27]$$

Now $\nabla\cdot\boldsymbol{\sigma}_1$ is replaced by the appropriate distribution of singularities, $\mathbf{E}\cdot\nabla\delta(\mathbf{x}-\boldsymbol{\xi})$. An integration by parts of this generalized function leads to the same functional of $\mathbf{e}^\infty(\boldsymbol{\xi})$. For example, Cox's (1969) exterior solution for a spherical drop in the linear field may be rewritten as a singularity solution so that

$$\nabla\cdot\boldsymbol{\sigma}_1 = -\frac{4}{3}\pi\mu_1 a^3 \frac{5\lambda+2}{\lambda+1}\left[1 + \frac{\lambda}{2(5\lambda+2)}a^2\nabla^2\right]\mathbf{E}\cdot\nabla\delta(\mathbf{x}). \qquad [28]$$

The integration by parts of [27] yields the Faxén relation for the stresslet as

$$\mathbf{S} = \frac{4}{3}\pi\mu_1 a^3 \frac{5\lambda+2}{\lambda+1}\left[1 + \frac{\lambda}{2(5\lambda+2)}a^2\nabla^2\right]\mathbf{e}^\infty(\mathbf{0}). \qquad [29]$$

The Faxén law for the force is obtained in the same manner from

$$\mathbf{U}\cdot\mathbf{F} = \int_{V_p^+} \mathbf{v}^\infty\cdot(\nabla\cdot\boldsymbol{\sigma}_1)\,dV, \qquad [30]$$

where now $\boldsymbol{\sigma}_1$ is the stress field of the exterior solution for the viscous drop in the uniform stream $\mathbf{U}$. For a spherical drop the specific result is the dual relation cited in section 1.

Acknowledgements—This material is based upon work supported by the National Science Foundation under Grant Nos CBT-8404451 and CBT-8451056 with matching funds from Kimberly-Clark and Dow-Corning. The Mathematics Research Center is supported by the U.S. Army under Contract DAAG29-80-C0041 and by the NSF under Grant DMS8210950 Mod. 4.

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